Utility Option Pricing Model (UOPM)
The Multi-State Asset Price Model

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In complete markets, options are valued using the Black-Scholes Option Pricing Model (BSOPM). In this white paper we will value put and call options in an incomplete market using the proprietary Utility Option Pricing Model (UOPM). To that end we will extend the two-state asset price model to the multi-state asset price model and work through the following hypothetical problem...

Our Hypothetical Problem

We are given the following model assumptions from the previous white paper...

Table 1: Model Assumptions

Definition	Value
Share price at time zero ($)	25.00
Option exercise price at time T ($)	27.50
Risk-free interest rate (%)	4.25
Cost of capital (i.e. discount rate) (%)	9.50
Dividend yield (%)	2.50
Share return volatility (%)	18.00
Risk aversion coefficient (#) (see note)	2.80
Option term in years (#)	3.00

Table 2: Model Notes

Note: The value of the coefficient of risk aversion is calibrated to the market via an iterative process. This calibration was done in the white paper **Utility Functions: Case Study - Investor Risk Aversion Coefficient** - November, 2023.

Question 1: What is UOPM call option value at time zero?
Question 2: What is UOPM put option value at time zero?
Question 3: Compare BSOPM (complete market) values with UOMP (incomplete market) values.

The Two-State Asset Price Model

Using the two-state asset pricing model, we defined the variable S_T to be random share price at time T, the variable μ to be continuous-time return drift, the variable σ to be continuous-time return volaility, and the variable z to be a random variable with value one or negative one at time T. The equation for random share price at time T as a function of known share price at time zero is... [1]

$$S_T = S_0 \,\mathrm{Exp}\left\{\mu\, T + \sigma\sqrt{T}\, z\right\} \text{ ...where... } z \in \{-1, +1\} \tag{1}$$

We will define the variable κ to be the risk-adjusted cost of capital (i.e. discount rate) and the variable ϕ to be the dividend yield. Using these definitions we can rewrite share price Equation (1) above as... [1]

$$S_T = S_0 \,\mathrm{Exp}\left\{\left(\kappa - \phi\right) T + \sigma\sqrt{T}\, z\right\} \text{ ...where... } \mu = \kappa - \phi \tag{2}$$

Using share price Equation (2) above, the two possible random share prices at time T are... [1]

$$S(U)_T = S_0 \,\mathrm{Exp}\left\{\left(\kappa - \phi\right) T + \sigma\sqrt{T}\right\} \text{ ...and... } S(D)_T = S_0 \,\mathrm{Exp}\left\{\left(\kappa - \phi\right) T - \sigma\sqrt{T}\right\} \tag{3}$$

We will define the variables C_T to be the value of a call option at time T, the variable P_T to be the value of a put option at time T, and the variable X_T to be the option's exercise price at time T. The equation for the values of a call and a put option at time T are... [1]

$$C_T = \text{Max}\Big[S_T - X_T, 0\Big] \text{ ...and... } P_T = \text{Max}\Big[X_T - S_T, 0\Big] \tag{4}$$

The Multi-State Asset Price Model

The two-state asset price model assumes that there are two possible share prices at time T. The multi-state model assumes that there are an infinite number of possible share prices at time T. [4]

$$S_T = S_0 \,\text{Exp}\left\{\left(\kappa - \phi - \frac{1}{2}\,\sigma^2\right) T + \sigma \sqrt{T}\, z\right\} \text{ ...where... } z \sim N\Big[0, 1\Big] \tag{5}$$

With the two-state model the random variable z can have two possible values, positive one or negative one. With the multi-state model the random variable z is normally-distributed with mean zero and variance one such that the number of possible share prices at time T is infinite.

We will define the variable m to be return mean and the variable v to be return variance. Using Equation (5) above as our guide, the equations for return mean and variance are...

$$m = \left(\kappa - \phi - \frac{1}{2}\,\sigma^2\right) T \text{ ...and... } v = \sigma^2\, T \tag{6}$$

Using the definitions in Equation (6) above, we can rewrite Equation (5) above as...

$$S_T = S_0 \,\text{Exp}\Big\{\theta\Big\} \text{ ...where... } \theta \sim N\Big[m, v\Big] \tag{7}$$

Given that share returns are normally distributed with mean m and variance v, the equation for the probability that share return will be between a and b is..

$$\text{Prob}\Big[a < \theta < b\Big] = \int\limits_a^b \sqrt{\frac{1}{2\,\pi\, v}}\, \text{Exp}\left\{-\frac{1}{2\,v}\,(\theta - m)^2\right\} \delta\theta \tag{8}$$

Using Equation (7) above, the equation for expected share price at time T is... [4]

$$\mathbb{E}\Big[S_T\Big] = \int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi\, v}}\, \text{Exp}\left\{-\frac{1}{2\,v}\,(\theta - m)^2\right\} S_0\, \text{Exp}\Big\{\theta\Big\}\, \delta\theta = S_0\, \text{Exp}\Big\{\Big(\kappa - \phi\Big) T\Big\} \tag{9}$$

Exponential Utility

We will define the variable O_i to be the scaled option payoff at time T given the i'th state of the world, the variable $U(O_i)$ to be the utility of the scaled option payoff, and the variable λ to be the average investor's coefficient of risk aversion. The equation for the utility of the scaled option payoff at time T is... [3]

$$U(O_i) = 1 - \text{Exp}\Big\{-\lambda\, O_i\Big\} \text{ ...where... } O_i = \text{Option payoff at time } T \Big/ \text{Share price at time zero} \tag{10}$$

We will define the variable p_i to be the probability of the i'th state of the world at time T and the variable n to be the number of possible states of the world at time T . Using Equation (10) above, the equation for the expected utility of the scaled option payoff is... [3]

$$\mathbb{E}\Big[U(\text{O})\Big] = \sum_{i=1}^{n} p_i\, U(O_i) = 1 - \sum_{i=1}^{n} p_i\, \text{Exp}\Big\{-\lambda\, O_i\Big\} \tag{11}$$

We will define the variable CE to be the option payoff certainty equivalent and the variable α to be the risk-free rate. The certainty equivalent is the amount certain that we will pay today for the random (i.e. risky) option payoff at time T. The amount that we will pay today is such that the utility of the certainty equivalent equals the present value of Equation (11) above. This statement in equation form is...

$$U(CE) = \mathbb{E}\Big[U(W)\Big]\,\text{Exp}\Big\{-\alpha\,T\Big\} \tag{12}$$

We will define the variable O_0 to be option value at time zero. Using the equations above, the value of the option at time zero is... [1]

$$O_0 = -S_0\,\ln\bigg(1 - \mathbb{E}\Big[U(O)\Big]\bigg)\,\text{Exp}\Big\{-\alpha\,T\Big\}\,\lambda^{-1} \tag{13}$$

The Answers To Our Hypothetical Problem

Using the data in Table 1 above, the values for the variables that we will need for our model are...

$$\begin{aligned} &S_0 = 25.00,\;\; X_T = 27.50,\;\; T = 3.00,\;\; \lambda = 0.2800,\;\; \sigma = 0.1800 \\ &\alpha = \ln(1 + 0.0425) = 0.0416,\;\; \kappa = \ln(1 + 0.0950) = 0.0908,\;\; \phi = \ln(1 + 0.0250) = 0.0247 \end{aligned} \tag{14}$$

Using Equations (5) and (14) above, return mean and variance over the time interval $[0, 3]$ are...

$$m = \left(0.0908 - 0.0247 - \frac{1}{2} \times 0.1800^2\right) \times 3.00 = 0.1496 \text{ ...and... } v = 0.1800^2 \times 3.00 = 0.0972 \tag{15}$$

The table for our analysis is...

Bucket	Random Return			Probability		Share	Option Payoff		Expected Utility	
Number	From	To	Average	Cumul	Increm	Price	Call	Put	Call	Put
1	-1.0975	-0.9977	-1.0476	0.0001	0.0001	8.77	0.00	18.73	0.0000	0.0001
2	-0.9977	-0.8980	-0.9478	0.0002	0.0002	9.69	0.00	17.81	0.0000	0.0001
3	-0.8980	-0.7982	-0.8481	0.0007	0.0005	10.71	0.00	16.79	0.0000	0.0004
4	-0.7982	-0.6984	-0.7483	0.0020	0.0013	11.83	0.00	15.67	0.0000	0.0011
5	-0.6984	-0.5987	-0.6485	0.0052	0.0032	13.07	0.00	14.43	0.0000	0.0026
6	-0.5987	-0.4989	-0.5488	0.0125	0.0073	14.44	0.00	13.06	0.0000	0.0056
7	-0.4989	-0.3991	-0.4490	0.0274	0.0149	15.96	0.00	11.54	0.0000	0.0108
8	-0.3991	-0.2994	-0.3492	0.0548	0.0274	17.63	0.00	9.87	0.0000	0.0183
9	-0.2994	-0.1996	-0.2495	0.1003	0.0455	19.48	0.00	8.02	0.0000	0.0270
10	-0.1996	-0.0998	-0.1497	0.1685	0.0683	21.52	0.00	5.98	0.0000	0.0333
11	-0.0998	-0.0001	-0.0499	0.2611	0.0926	23.78	0.00	3.72	0.0000	0.0315
12	-0.0001	0.0997	0.0498	0.3745	0.1134	26.28	0.00	1.22	0.0000	0.0145
13	0.0997	0.1995	0.1496	0.5000	0.1255	29.03	1.53	0.00	0.0198	0.0000
14	0.1995	0.2992	0.2494	0.6255	0.1255	32.08	4.58	0.00	0.0504	0.0000
15	0.2992	0.3990	0.3491	0.7389	0.1134	35.45	7.95	0.00	0.0668	0.0000
16	0.3990	0.4988	0.4489	0.8315	0.0926	39.16	11.66	0.00	0.0675	0.0000
17	0.4988	0.5985	0.5486	0.8997	0.0683	43.27	15.77	0.00	0.0566	0.0000
18	0.5985	0.6983	0.6484	0.9452	0.0455	47.81	20.31	0.00	0.0408	0.0000
19	0.6983	0.7981	0.7482	0.9726	0.0274	52.83	25.33	0.00	0.0258	0.0000
20	0.7981	0.8978	0.8479	0.9875	0.0149	58.37	30.87	0.00	0.0144	0.0000
21	0.8978	0.9976	0.9477	0.9948	0.0073	64.50	37.00	0.00	0.0072	0.0000
22	0.9976	1.0974	1.0475	0.9980	0.0032	71.26	43.76	0.00	0.0032	0.0000
23	1.0974	1.1971	1.1472	0.9993	0.0013	78.74	51.24	0.00	0.0013	0.0000
24	1.1971	1.2969	1.2470	0.9998	0.0005	87.00	59.50	0.00	0.0005	0.0000
25	1.2969	1.3967	1.3468	0.9999	0.0002	96.13	68.63	0.00	0.0002	0.0000
Total	–	–	–	–	0.9999	–	–	–	0.3544	0.1453

Given that there is no closed-form solution to Equation (11) above, we will solve that equation via numerical integration. Using Equation (15) above, the valuation integral will be in the following bounds...

$$\text{Lower bound} = 0.1496 - \sqrt{0.0972} \times 4.00 = -1.0975 \text{ ...and... } \text{Upper bound} = 0.1496 + \sqrt{0.0972} \times 4.00 = 1.3967 \tag{16}$$

Example: Row 20

Using Equations (8) and (15) above, the cumulative probability of realizing return = 0.8479 is...

Cumulative probability = NormDist(0.8479, 0.1496, Sqr(0.0972), True) = 0.9875 (Excel function)

Using Equations (7) and (14) above, share price given this state of the world is...

$$S_3 = 25.00 \times \text{Exp}\left\{0.8479\right\} = 58.37 \tag{17}$$

Using Equations (4), (14), and (17) above, option payoffs given this state of the world are...

$$C_3 = \text{Max}\left[58.37 - 27.50, 0\right] = 30.87 \text{ ...and... } P_3 = \text{Max}\left[25.00 - 58.37, 0\right] = 0.00 \tag{18}$$

Using Equations (10) and (14) above, the expected utility of the call payoff (put payoff is zero) is...

$$U(\text{Call payoff}) = \left(1 - \text{Exp}\left\{-2.80 \times \frac{30.87}{25.00}\right\}\right) \times 0.0149 = 0.0144 \tag{19}$$

Question 1: What is call option value at time zero?

Using Equation (13) and our table for analysis above, the answer to the question is...

$$O_0 = -25.00 \times \ln\left(1 - 0.3544\right) \times \text{Exp}\left\{-0.0416 \times 3.00\right\} \times 2.80^{-1} = 3.45 \tag{20}$$

Question 2: What is put option value at time zero?

Using Equation (13) and our table for analysis above, the answer to the question is...

$$O_0 = -25.00 \times \ln\left(1 - 0.1453\right) \times \text{Exp}\left\{-0.0416 \times 3.00\right\} \times 2.80^{-1} = 1.24 \tag{21}$$

Question 3: Compare Black-Scholes values (BSOPM) with UOMP values.

Using Equations (20) and (21) above, the answer to the question is...

Description	UOPM	BSOPM	Difference
Call option	3.45	2.44	41%
Put option	1.24	3.50	-65%

Note: Most of the difference between model values lies in the fact that the Black-Scholes model uses the risk-neutral probability distribution, which applies more weight to bad outcomes (stock price decreases/BSOPM put option price increases) and less weight to good outcomes (stock price increases/BSOPM call option price decreases).

References

[1] Gary Schurman, *UOPM - Two-State Model - Option Pricing in Incomplete Markets,* December, 2023.

[2] Gary Schurman, *Case Study - Investor Risk Aversion Coefficient,* October, 2023.

[3] Gary Schurman, *The Exponential Utility Funtion,* October, 2023.

[4] Gary Schurman, *Brownian Motion - Introduction to Stochastic Calculus,* February, 2012.